Exercise 65

Is there a number a such that

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

Solution

Factor the denominator.

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2} = \lim_{x \to -2} \frac{3x^2 + ax + (a + 3)}{(x + 2)(x - 1)}$$

Plugging in $x = -2$ makes the denominator zero, so the goal is to write the numerator as $(x + 2)f(x)$ so that the $(x + 2)$ factor in the denominator cancels out.

$$\begin{aligned}
\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2} &= \lim_{x \to -2} \frac{(x + 2)f(x)}{(x + 2)(x - 1)} \\
&= \lim_{x \to -2} \frac{f(x)}{x - 1} \\
&= \frac{\lim_{x \to -2} f(x)}{\lim_{x \to -2} (x - 1)} \\
&= \frac{\lim_{x \to -2} f(x)}{-2 - 1} \\
&= -\frac{1}{3} \lim_{x \to -2} f(x)
\end{aligned}$$

Figure out what $f(x)$ is by using long division.

$$(x + 2)f(x) = 3x^2 + ax + (a + 3)$$

$$f(x) = \frac{3x^2 + ax + (a + 3)}{x + 2}$$

$$x + 2 \,\overline{\big)\, 3x^2 + ax + (a + 3)}$$

Multiplying x by $3x$ gives the $3x^2$ term.

$$\begin{array}{r} 3x \\ x + 2 \,\overline{\big)\, 3x^2 + ax + (a + 3)} \end{array}$$

Multiply the divisor by $3x$ and subtract the result from the dividend.

$$\begin{array}{r|l} & 3x \\ \hline x+2 & 3x^2 + ax + (a+3) \\ & -\left(3x^2 + 6x\right) \end{array}$$

Do the subtraction.

$$\begin{array}{r|l} & 3x \\ \hline x+2 & 3x^2 + ax + (a+3) \\ & -\left(3x^2 + 6x\right) \\ \cline{2-2} & \quad (a-6)x \end{array}$$

Bring down the next term from the dividend.

$$\begin{array}{r|l} & 3x \\ \hline x+2 & 3x^2 + ax + (a+3) \\ & -\left(3x^2 + 6x\right) \quad \downarrow \\ \cline{2-2} & \quad (a-6)x + (a+3) \end{array}$$

Multiplying x by $(a-6)$ gives the $(a-6)x$ term.

$$\begin{array}{r}
3x + (a-6) \\
x+2 \,\big)\, \overline{3x^2 + ax + (a+3)} \\
-(3x^2 + 6x) \\
\hline
(a-6)x + (a+3)
\end{array}$$

Multiply the divisor by $(a-6)$ and subtract the result from the modified dividend.

$$\begin{array}{r}
3x + (a-6) \\
x+2 \,\big)\, \overline{3x^2 + ax + (a+3)} \\
-(3x^2 + 6x) \\
\hline
(a-6)x + (a+3) \\
-((a-6)x + 2(a-6))
\end{array}$$

Do the subtraction.

$$\begin{array}{r}
3x + (a-6) \\
x+2 \,\big)\, \overline{3x^2 + ax + (a+3)} \\
-(3x^2 + 6x) \\
\hline
(a-6)x + (a+3) \\
-((a-6)x + 2(a-6)) \\
\hline
-a + 15
\end{array}$$

There are no other terms to bring down, so $-a+15$ is the remainder.

$$f(x) = \frac{3x^2 + ax + (a+3)}{x+2} = 3x + (a-6) + \frac{-a+15}{x+2}$$

Since we want $f(x)$ to be defined at $x=-2$, set $a=15$ to eliminate the remainder.

$$f(x) = 3x + 9$$

Therefore,

$$\begin{aligned}
\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2} = -\frac{1}{3} \lim_{x \to -2} f(x) = -\frac{1}{3} \lim_{x \to -2} (3x+9) &= -\frac{1}{3}(3) \\
&= -1.
\end{aligned}$$